A

Solutions to Exercises

A.1 Chapter 1 Solutions

Solution 1 Color JPEG compressed images are typically 5–50 times smaller than they would be if stored "naively," so the ratio of naively-stored to JPEG-stored might range from a low of 0.02–0.2.

Solution 2

- From Euler's identity we have (for x real)

$$\begin{aligned} e^{-ix} &= \cos(x) + i\sin(-x) \\ &= \cos(x) - i\sin(x) \\ &= \overline{\cos(x) + i\sin(x)} \\ &= \overline{e^{ix}}. \end{aligned}$$

- If $e^{2\pi ix} = 1$ then $\cos(2\pi x) + i\sin(2\pi x) = 1$. We conclude that $\sin(2\pi x) = 0$, which forces $2\pi x = 2\pi k$ for some integer k, that is, $x = k$. Conversely, if x is an integer k then $e^{2\pi ix} = e^{2\pi ik} = \cos(2\pi k) + i\sin(2\pi k) = 1$.

Solution 3 The eighth roots of unity are the numbers $e^{2\pi ik/8}$ where $0 \le k \le 7$, and are equal to

$$1,\quad \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,\quad i,\ -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,\ -1,\ -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,\ -i,\quad \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

(moving counterclockwise around the unit circle).

Solution 4 The mth component of $\mathbf{E}_{N,k}$ is given by equation (1.23) and is $\mathbf{E}_{N,k}(m) = \exp(2\pi ikm/N)$. But $(\exp(2\pi ikm/N))^N = \exp(2\pi ikm) = 1$ since k and m are integers (refer to Exercise 2).

Discrete Fourier Analysis and Wavelets: Applications to Signal and Image Processing, Second Edition.
S. Allen Broughton and Kurt Bryan.

Companion Website: www.wiley.com/go/Broughton/Discrete_Fourier_Analysis_and_Wavelets

Solution 5 We have from Euler's identity

$$\begin{aligned} x(t) = a\cos(\omega t) + b\sin(\omega t) &= \frac{a}{2}(e^{i\omega t}) + e^{-i\omega t}) + \frac{b}{2i}(e^{i\omega t}) - e^{-i\omega t}) \\ &= \frac{a}{2}(e^{i\omega t} + e^{-i\omega t}) - \frac{ib}{2}(e^{i\omega t}) - e^{-i\omega t}) \\ &= \frac{1}{2}(a - ib)e^{i\omega t} + \frac{1}{2}(a + ib)e^{-i\omega t} \end{aligned}$$

(since $1/i = -i$). Comparison to $x(t) = ce^{i\omega t} + de^{-i\omega t}$ shows that $c = \frac{a-ib}{2}, d = \frac{a+ib}{2}$. These last two linear equations are easily solved for a and b to yield $a = c + d, b = i(c - d)$.

Solution 6

(a) The sampled versions are

$$\mathbf{x} = \langle 0, 0.325, 0.65, 0.975\rangle$$
$$\mathbf{y} \approx \langle 0, 0.382, 0.707, 0.924\rangle$$

Of course the result of sampling $x(t) + y(t)$ is just $\mathbf{x} + \mathbf{y} \approx \langle 0, 0.708, 1.36, 1.90\rangle$.

(b) We find

$$q(\mathbf{x}) = \langle 0, 0, 1, 1\rangle$$
$$q(\mathbf{y}) = \langle 0, 0, 1, 1\rangle$$
$$q(\mathbf{x} + \mathbf{y}) = \langle 0, 1, 1, 2\rangle.$$

Then $q(\mathbf{x}) + q(\mathbf{y}) \neq q(\mathbf{x} + \mathbf{y})$. Also, $q(2\mathbf{x}) = \langle 0, 1, 1, 2\rangle$ which is not $2q(\mathbf{x})$.

Solution 7 Let $x(t) = \sin(\pi t/2)$ on $0 \le t \le 1$ (note $0 \le x(t) \le 1$ on this interval). Suppose we sample $x(t)$ at times $t = 0.2, 0.4, 0.6, 0.8$.

(a) We find values

0.3090169944, 0.5877852525, 0.8090169943, 0.9510565165

(b) Rounding to the nearest multiple of 0.25 yields 0.25, 0.50, 0.75, 1.00. This distortion is 8.534×10^{-3}.

(c) Rounding to the nearest multiple of 0.25 yields 0.3, 0.6, 0.8, 1.00. This distortion is 1.354×10^{-3}. Rounding to the nearest multiple of 0.05 yields 0.30, 0.60, 0.80, 0.95. This distortion is 1.56×10^{-4}.

(d) The codebook here consists of some range of multiples of h. The dequantization map is simply $\tilde{q}(k) = kh$.

Solution 8 Yes, this is a vector space. It is clearly closed under addition and scalar multiplication. Addition commutes, is associative, the zero vector is the zero polynomial, and all the other rules of Definition 1.1 are straightforward to verify.

Solution 9 This is not a vector space—it is not closed under addition or scalar multiplication, for example, if

$$\int_0^1 f(x)\,dx = 3$$

then the function $2f$ satisfies

$$\int_0^1 2f(x)\,dx = 6$$

and is not in the set.

Solution 10 $\mathbb{R}^n$ is NOT a subspace of $\mathbb{C}^n$, if $\mathbb{C}^n$ is taken as a vector space over $\mathbb{C}$, for we do not have closure under scalar multiplication. For example, in the case $n = 2$, we can take $\mathbf{v} = \langle 1, 2\rangle \in \mathbb{R}^2$ and scalar $c = i$. But $c\mathbf{v} = \langle i, 2i\rangle$ is not in $\mathbb{R}^2$.

Solution 11 With the operations as defined, we clearly have closure in addition and scalar multiplication. The commutativity and associativity of addition follow from the same properties for the reals numbers. The zero vector is just the element

$$\mathbf{0} = (0, 0, 0, \ldots).$$

The inverse of any element $\mathbf{x} = (x_0, x_1, \ldots)$ is

$$(-x_0, -x_1, -x_2, \ldots).$$

Properties (e)–(h) in Definition 1.1 follow immediately from component-by-component application of the corresponding properties for real numbers.

Solution 12 As remarked in Example 1.8, we have $(p + q)^2 \le 2p^2 + 2q^2$ for any real numbers p and q. As a result if $\mathbf{x} = (x_0, x_1, \ldots)$ and $\mathbf{y} = (y_0, y_1, \ldots)$ are elements of $L^2(\mathbb{N})$ then $(x_k + y_k)^2 \le 2x_k^2 + 2y_k^2$. Sum both sides to conclude that

$$\sum_{k=0}^{\infty}(x_k + y_k)^2 \le 2\left(\sum_{k=0}^{\infty} x_k^2 + \sum_{k=0}^{\infty} y_k^2\right) < \infty$$

so the sum $\mathbf{x} + \mathbf{y}$ is in $L^2(\mathbb{N})$ and we have closure under addition. Clearly $\sum_k (cx_k)^2 = c^2 \sum_k x_k^2$, so we have closure under scalar multiplication. All other properties are verified exactly just as for Exercise 11.

If $\sum_k x_k^2 < \infty$ then the x_k themselves must limit to zero (a basic property of convergent series). Thus, for example, for all k sufficiently large, say $k \ge N$ for some N, we have $x_k^2 \leq 1$. Let $M = \max(1, x_0^2, x_1^2, \ldots, x_{N-1}^2)$. Then we have $|x_k| \leq \sqrt{M}$ for all k. Thus, $L^2(\mathbb{N})$ is a subset of $L^\infty(\mathbb{N})$ which is closed under addition and scalar multiplication, hence a subspace.

Solution 13

(a) If there were two "zero vectors" $\mathbf{0}_1$ and $\mathbf{0}_2$ then we must have $\mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_1$ from property (iii/c) in Definition 1.1. But we also must have $\mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2$. Comparison of the last two equations shows that $\mathbf{0}_1 = \mathbf{0}_2$.

(b) That the left and right sides in line 1 are equal follows from (iii/g). The left side of line 2 equals the left side of 1 since $1 + 0 = 1$ in the reals, and the right side of line 2 equals the right side of line 1 due to property (iii/h). The left side of line 3 equals the left side of line 2 due to (iii/h). Line 4 follows by adding $-u$ to both sides of line 3 and invoking commutativity and associativity (iii/a, iii/b). Line 5 follows from the definition of the additive inverse $-u$, and line 6 follows from the definition of $\mathbf{0}$, the zero vector.

(c) Many ways to proceed. Start with $\mathbf{u} + \mathbf{v} = \mathbf{0}$, which is equivalent to

$$\mathbf{u} + \mathbf{v} = (1 + (-1))\mathbf{u}$$

since from part (b) we know $0\mathbf{u} = \mathbf{0}$ (and $1 + (-1) = 0$ in the reals!). The above equation leads to (distribute according to property (iii/f)) $\mathbf{u} + \mathbf{v} = \mathbf{u} + (-1)\mathbf{u}$. Add $-\mathbf{u}$ to both sides and get the desired conclusion, $\mathbf{v} = (-1)\mathbf{u}$.

Solution 14 The vectors are, for the case $N = 2$,

$$\mathbf{E}_{2,-2} = \mathbf{E}_{2,0} = \mathbf{E}_{2,2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{E}_{2,-1} = \mathbf{E}_{2,1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The vectors are, for the case $N = 2$,

$$\mathbf{E}_{3,-2} = \mathbf{E}_{3,1} = \begin{bmatrix} 1 \\ \dfrac{-1 + i\sqrt{3}}{2} \\ \dfrac{-1 - i\sqrt{3}}{2} \end{bmatrix}, \quad \mathbf{E}_{3,-1} = \mathbf{E}_{3,2} = \begin{bmatrix} 1 \\ \dfrac{-1 - i\sqrt{3}}{2} \\ \dfrac{-1 + i\sqrt{3}}{2} \end{bmatrix}, \quad \mathbf{E}_{3,0} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution 15

(a) We have $|f(t)| = |a||e^{i\omega t}| = |a|$ (since it follows from Euler's identity that $|e^{i\omega t}| = \sqrt{\sin^2(\omega t) + \cos^2(\omega t)} = 1$). By the same reasoning we have $|a| = |r||e^{i\theta}| = |r| = r$ (since $r > 0$). Then $|f(t)| = r$.

(b) From $f(t) = ae^{i\omega t}$ and $a = re^{i\theta}$ we have

$$f(t) = re^{i(\omega t + \theta)} = r\cos(\omega t + \theta) + ir\sin(\omega t + \theta).$$

Now, for example,

$$\cos(\omega t + \theta) = \cos(\omega(t + \theta/\omega))$$

so that the graph of $\cos(\omega t + \theta)$ is the graph of $\cos(\omega t)$ shifted a distance θ/ω to the left. This corresponds to a fraction θ/ω of one period. A similar consideration applies to $\sin(\omega t + \theta)$. We conclude that the graph of $ae^{i\omega t}$ is just $e^{i\omega t}$ shifted "to the left." This actually corresponds to advancing the signal in time (think about why—$f(t)$ will attain a given value θ/ω in advance of $e^{i\omega t}$).

Solution 16 We can write (using Euler's identity)

$$\begin{aligned}\cos(\alpha x)\cos(\beta y) &= \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}\frac{e^{i\beta y} + e^{-i\beta y}}{2}\\ &= \frac{1}{4}e^{i\alpha x}e^{i\beta y} + \frac{1}{4}e^{-i\alpha x}e^{i\beta y} + \frac{1}{4}e^{i\alpha x}e^{-i\beta y} + \frac{1}{4}e^{-i\alpha x}e^{-i\beta y}.\end{aligned}$$

Similarly,

$$\begin{aligned}\cos(\alpha x)\sin(\beta y) &= -\frac{i}{4}e^{i\alpha x}e^{i\beta y} - \frac{i}{4}e^{-i\alpha x}e^{i\beta y} + \frac{i}{4}e^{i\alpha x}e^{-i\beta y} + \frac{i}{4}e^{-i\alpha x}e^{-i\beta y},\\ \sin(\alpha x)\cos(\beta y) &= -\frac{i}{4}e^{i\alpha x}e^{i\beta y} + \frac{i}{4}e^{-i\alpha x}e^{i\beta y} - \frac{i}{4}e^{i\alpha x}e^{-i\beta y} + \frac{i}{4}e^{-i\alpha x}e^{-i\beta y},\\ \sin(\alpha x)\sin(\beta y) &= -\frac{1}{4}e^{i\alpha x}e^{i\beta y} + \frac{1}{4}e^{-i\alpha x}e^{i\beta y} + \frac{1}{4}e^{i\alpha x}e^{-i\beta y} - \frac{1}{4}e^{-i\alpha x}e^{-i\beta y}.\end{aligned}$$

Solution 17 The relation $\mathbf{E}_k = \mathbf{C}_k + i\mathbf{S}_k$ follows from Euler's identity, for the mth component of $\mathbf{E}_k$ is $\mathbf{E}_k(m) = e^{2\pi imk/N}$, which is $\cos(2\pi mk/N) + i\sin(2\pi mk/N) = \mathbf{C}_k(m) + i\mathbf{S}_k(m)$, where $\mathbf{C}_k(m)$ and $\mathbf{S}_k(m)$ are the mth components of $\mathbf{C}_k$ and $\mathbf{S}_k$, respectively.

The equation $\overline{\mathbf{E}_k} = \mathbf{C}_k - i\mathbf{S}_k$ follows from $\overline{\mathbf{E}_k(m)} = \overline{e^{2\pi imk/N}} = e^{-2\pi imk/N} = \cos(2\pi mk/N) - i\sin(2\pi mk/N) = \mathbf{C}_k(m) - i\mathbf{S}_k(m)$. The relations $\mathbf{C}_k = \frac{1}{2}(\mathbf{E}_k + \overline{\mathbf{E}_k})$ and $\mathbf{S}_k = \frac{1}{2i}(\mathbf{E}_k - \overline{E_k})$ are similar and follow from component-by-component application of $\mathbf{C}_k(m) = \frac{1}{2}(\mathbf{E}_k(m) + \overline{\mathbf{E}_k(m)}$ and $\mathbf{C}_k(m) = \frac{1}{2i}(\mathbf{E}_k(m) - \overline{\mathbf{E}_k(m)}$ (which themselves are consequences of equation (1.13)).

The final two relations are similar and consequences of component-by-component application of $\mathbf{C}_k(m) = \mathrm{Re}(\mathbf{E}_k(m))$ and $\mathbf{S}_k(m) = \mathrm{Im}(\mathbf{E}_k(m))$.

Solution 18 If we treat $\mathbf{E}_{m,k}$ as an $m \times 1$ matrix, the row ith column r entry is

$$\mathbf{E}_{m,k}(r, 1) = \exp(2\pi irk/m).$$

The quantity $\mathbf{E}^T_{n,l}$ is a $1 \times n$ matrix with row 1 column s entry

$$\mathbf{E}^T_{n,l}(1, s) = \exp(2\pi isl/n).$$

The definition of matrix multiplication dictates that the row r column s entry of the product $\mathbf{E}_{m,k}\mathbf{E}_{n,l}^T$ is given by

$$\begin{aligned}(\mathbf{E}_{m,k}\mathbf{E}_{n,l}^T)(r,s) &= \sum_{p=1}^{1} \mathbf{E}_{m,k}(r,p)(\mathbf{E}_{n,l}^T)(p,s)\\ &= \mathbf{E}_{m,k}(r,1)(\mathbf{E}_{n,l}^T)(1,s)\\ &= \exp(2\pi irk/m)\exp(2\pi isl/n)\\ &= \exp(2\pi i(rk/m + sl/n)).\end{aligned}$$

(Note the sum above is trivial—there is only one term.) The last equation is precisely equation (1.25).

Solution 19

(a) We have, from the given information,

$$\begin{aligned}g(t) &= \exp(2\pi i(px(t) + qy(t)))\\ &= \exp(2\pi i[(px_0 + qy_0) + (pu_1 + qu_2)t])\\ &= \exp(2\pi i(px_0 + qy_0))\exp((pu_1 + qu_2)t)\\ &= A\exp((\mathbf{u}\cdot\mathbf{v})t),\end{aligned}$$

where $A = \exp(2\pi i(px_0 + qy_0))$. We have $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta) = \|\mathbf{v}\| \cos(\theta)$ since $\|\mathbf{u}\| = 1$. All in all this yields

$$g(t) = Ae^{2\pi i\|\mathbf{v}\|\cos(\theta)t}. \tag{A.1}$$

(b) If L is parallel to the unit vector $\mathbf{u}$ and orthogonal to $\mathbf{v}$ then $\mathbf{u}\cdot\mathbf{v} = 0$, or equivalently, $\cos(\theta) = 0$. Thus, $g(t)$ from above is identically equal to the constant A along L, and so is $f(t)$.

(c) The function $e^{2\pi i\|\mathbf{v}\|\cos(\theta)t}$ (and hence f in the direction $\mathbf{u}$) goes through

$$F = \|\mathbf{v}\|\cos(\theta) = (p^2 + q^2)^{1/2}\cos(\theta)$$

cycles per unit change in t (the function $e^{2\pi iFt}$ goes through F cycles per unit change in t).

(d) The θ values that maximizes F from above is $\theta = 0$, corresponding to motion in the direction $\mathbf{v}$ (orthogonal to the line L in part (b)). In this direction the function f oscillates at $\sqrt{p^2 + q^2}$ cycles per unit motion. In part (b) we had $\theta = \pi/2$.

(e) The peak-to-peak distance is just $1/F$ with $\theta = 0$, that is, $1/\sqrt{p^2 + q^2}$. Compare this to the images in Figure 1.7.

Solution 20 The vectors $\mathbf{E}_{6,k}$ are, in the order $k = 0, 1, 2, 3, 4, 5$ (and components indexed starting from 0),

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \\ -1 \\ \frac{-1-i\sqrt{3}}{2} \\ \frac{1-i\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \\ 1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \\ 1 \\ \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \\ -1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{1+i\sqrt{3}}{2} \end{bmatrix}.$$

We already know the vectors are periodic with period 6 beyond this range. With this range, we can see that $\mathbf{E}_{6,1} = \overline{\mathbf{E}_{6,5}}$ and $\mathbf{E}_{6,2} = \overline{\mathbf{E}_{6,4}}$. See the solution to the next exercise.

Solution 21 The mth component of $\mathbf{E}_{N,k}$ is

$$\mathbf{E}_{N,k}(m) = \exp(2\pi ikm/N)$$

and the mth component of $\mathbf{E}_{N,N-k}$ is

$$\mathbf{E}_{N,N-k}(m) = \exp(2\pi i(N-k)m/N) = \exp(2\pi im)\exp(-2\pi ikm/N) = \overline{\mathbf{E}_{N,k}}$$

since $\exp(2\pi im) = 1$ and $\overline{\exp(2\pi ikm/N)} = \exp(-2\pi ikm/N)$.

Solution 22 The row r column s entry for $\mathcal{E}_{m,n,k,l}$ is

$$\mathcal{E}_{m,n,k,l}(r,s) = \exp(2\pi i(rk/m + sl/n))$$

from equation (1.25). Then, for example,

$$\begin{aligned} \mathcal{E}_{m,n,k+m,l}(r,s) &= \exp(2\pi i((k+m)r/m + ls/n)) \\ &= \exp(2\pi ikr)\exp(2\pi i(rk/m + sl/n)) \\ &= \mathcal{E}_{m,n,k,l}(r,s). \end{aligned}$$

Component-by-component application of the above shows $\mathcal{E}_{m,n,k,l} = \mathcal{E}_{m,n,k+m,l}$, so $\mathcal{E}_{m,n,k,l}$ is periodic in k with period m. Similar analysis shows $\mathcal{E}_{m,n,k,l} = \mathcal{E}_{m,n,k,l+n}$.

As an example of a conjugate aliasing relation, we have

$$\begin{aligned} \mathcal{E}_{m,n,m-k,n-l}(r,s) &= \exp(2\pi i((m-k)r/m + (n-l)s/n)) \\ &= \exp(2\pi i(r+s))\exp(2\pi i(-rk/m - sl/n)) \\ &= \overline{\mathcal{E}_{m,n,k,l}(r,s)}. \end{aligned}$$

Component-by-component application of the above shows $\mathcal{E}_{m,n,k,l} = \overline{\mathcal{E}_{m,n,m-k,n-l}}$.

Solution 23

(a) One can easily compute that $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$. Note also that $\|\mathbf{v}_1\|^2 = 2$, $\|\mathbf{v}_1\|^2 = 3$, $\|\mathbf{v}_3\|^2 = 6$.

(b) Here $\mathbf{w} \cdot \mathbf{v}_1 = 7, \mathbf{w} \cdot \mathbf{v}_2 = 6, \mathbf{w} \cdot \mathbf{v}_3 = 9$. Thus (using the norms from part (a)),

$$\mathbf{w} = \frac{7}{2}\mathbf{v}_1 + 2\mathbf{v}_2 + \frac{3}{2}\mathbf{v}_3.$$

(c) The rescaled vectors are (in row form)

$$\mathbf{u}_1 = (1/\sqrt{2}, 1/\sqrt{2}, 0), \quad \mathbf{u}_2 = (-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}),$$
$$\mathbf{u}_3 = (1/\sqrt{6}, -1/\sqrt{6}, 2/\sqrt{6}).$$

(d) In this case we find

$$\mathbf{w} = \frac{7\sqrt{2}}{2}\mathbf{v}_1 + 2\sqrt{3}\mathbf{v}_2 + \frac{3\sqrt{6}}{2}\mathbf{v}_3.$$

(e) From part (d)

$$\left(\frac{7\sqrt{2}}{2}\right)^2 + (2\sqrt{3})^2 + \left(\frac{3\sqrt{6}}{2}\right)^2 = 3^2 + 4^2 + 5^2.$$

Both sides are, in fact, equal to 50.

Solution 24

(a) Here

$$\mathbf{w} = \frac{7}{4}\mathbf{E}_{4,0} + \frac{3+2i}{4}\mathbf{E}_{4,1} - \frac{9}{4}\mathbf{E}_{4,2} + \frac{3-2i}{4}\mathbf{E}_{4,3}.$$

(b) Each vector $\mathbf{E}_{4,k}$ has norm 2, so the rescaled vectors are just $\widetilde{\mathbf{E}}_{4,k} = \frac{1}{2}\mathbf{E}_{4,k}$.

(c) We find

$$\mathbf{w} = \frac{7}{2}\widetilde{\mathbf{E}}_{4,0} + (3/2 + i)\widetilde{\mathbf{E}}_{4,1} - \frac{9}{2}\widetilde{\mathbf{E}}_{4,2} + (3/2 - i)\widetilde{\mathbf{E}}_{4,3}.$$

(d) We should have

$$(7/2)^2 + |3/2 + i|^2 + (9/2)^2 + |3/2 - i|^2 = 1^2 + 5^2 + (-2)^2 + 3^2.$$

Both sides are, in fact, equal to 39.

Solution 25

(a) It is easy to check that $(\mathbf{v}, \mathbf{w})_d = \overline{(\mathbf{w}, \mathbf{v})_d}$. The conjugation is irrelevant here, and since $d_k v_k w_k = d_k w_k v_k$ the relevant sums defining $(\mathbf{v}, \mathbf{w})_d$ and $(\mathbf{w}, \mathbf{v})_d$ are the same.
We also have

$$\begin{aligned}(a\mathbf{u} + b\mathbf{v}, \mathbf{w})_d &= \sum_{k=1}^{n} d_k(au_k + bv_k)w_k \\ &= a\sum_{k=1}^{n} d_k u_k w_k + b\sum_{k=1}^{n} d_k v_k w_k \\ &= a(\mathbf{u}, \mathbf{w})_d + b(\mathbf{v}, \mathbf{w})_d.\end{aligned}$$

That $(\mathbf{v}, \mathbf{v})_d \geq 0$ follows since

$$(\mathbf{v}, \mathbf{v})_d = \sum_{k=1}^{n} d_k v_k^2$$

and each piece of the sum is nonnegative. Also, if $(\mathbf{v}, \mathbf{v})_d = 0$ have

$$\sum_{k=1}^{n} d_k v_k^2 = 0.$$

Since each term in the sum is nonnegative we conclude $d_k v_k^2 = 0$ for all k, forcing $v_k = 0$. Thus, $\mathbf{v} = \mathbf{0}$.
The norm defined by this inner product is

$$\|\mathbf{v}\|_d = \left(\sum_{k=1}^{n} d_k v_k^2\right)^{1/2}.$$

(b) In this case we have

$$(\mathbf{v}_1, \mathbf{v}_2)_d = (1)(2)(5) + (5)(1)(-2) = 0$$

so the vectors are orthogonal.

(c) The lengths are $\|\mathbf{v}_1\|_d = 3$ and $\|\mathbf{v}_2\|_d = 3\sqrt{5}$.

(d) Here

$$\mathbf{w} = \frac{7}{3}\mathbf{v}_1 - \frac{4}{3}\mathbf{v}_2.$$

Solution 26 As suggested, start with $\mathbf{v} = (\mathbf{v} - \mathbf{w}) + \mathbf{w}$ and take the norm of both sides, then apply the triangle inequality:

$$\begin{aligned}\|\mathbf{v}\| &= \|(\mathbf{v} - \mathbf{w}) + \mathbf{w}\| \\ &\leq \|(\mathbf{v} - \mathbf{w})\| + \|\mathbf{w}\|\end{aligned}$$